The trivial units property and the unique product property for torsion free groups

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I. The trivial units property

Definition

Let G be a torsion free group. Let R be a domain, i.e., a commutative ring with 1, and without zero divisors.

We say that G has the trivial units property for R (TUP_R) if the group ring $R[G]$ only has the trivial units:

all the units are of the form rg, where $r \in R^\times$ and $g \in G$.

- The group ring $R[G]$ consists of the formal sums $u = \sum_{g \in G} u_g g$ with coefficients $u_g \in R$, and only finitely many non-zero. The set $\{g \colon u_g \neq 0\}$ is called the support of u.
- Addition is "component-wise". The product is defined by $(\sum_{g \in G} u_g g)(\sum_{h \in G} v_h h) = \sum_{r \in G} w_r r$, where $w_r = \sum_{gh=r} u_g v_h$.
- $\alpha \in R[G]$ is a unit if $\exists \beta \, [\alpha\beta = \beta\alpha = 1e]$.

Unit conjecture: the story

Unit conjecture

Every torsion-free group G satisfies TUP_R, for each domain R.

- G. Higman's in his 1940 thesis posed the unit conjecture for $\mathbb{Z}$.
- Kaplansky's 1956 talk and 1970 paper posed it for general domains R.
- Unit conjecture was refuted by Gardam (2020) for $R[P]$, where $P =$ Hantzsche-Wendt group, $R = GF_2$.
- The conjecture is now refuted for each characteristic: Murray (2021) for GF_p, $p \geq 3$, and Gardam (2023) for $\mathbb{C}$, all for the same group P.
- Higman's original conjecture for $R = \mathbb{Z}$ remains open.

A nontrivial unit in $GF_2[P]$ supported on radius 6

Let $R = GF_2$, the field with two elements. Let

$$P = \langle a, b \mid b^{-1}a^2b = a^{-2}, a^{-1}b^2a = b^{-2} \rangle.$$

P is an extension of $\mathbb{Z}^3 \cong \langle a^2, b^2, (ab)^2 \rangle$ by $C_2 \times C_2$.

Theorem (Giles Gardam, Annals of Mathematics, 2021)

There is a nontrivial unit in $R[P]$.

- Elements of $R[G]$ can be identified with their supports.
- Gardam expressed the existence of a nontrivial unit as a Boolean satisfiability problem, with variables for membership in the supports. Then he used a SAT solver.
- In this way he found a unit α such that α and α^{-1} are supported on the ball around e of radius 6 in the Cayley graph of P. Both α and α^{-1} have support of size 21.

Example of SAT solver

- Tries to find satisfying assignment to a given Boolean formula, or reports that there is none.
- SAT solver processes conjunctive normal forms. Number $n \in \mathbb{N}^+$ denotes p_n, and $-n$ denotes $\neg p_n$.
- Each disjunction is a row, and rows are separated by 0.
- To query the CNF $(p_1 \vee \neg p_3) \wedge (p_2 \vee p_3 \vee \neg p_1)$, enter
  p cnf 3 2
  1 -3 0
  2 3 -1 0
- Minisat answers SATISFIABLE -1 -2 -3 0, meaning that setting all variables to FALSE yields a satisfying assignment.

A nontrivial unit in $GF_2[P]$ supported on radius 4

Write $\overline{x}$ for x^{-1}. Using the same strategy we found this unit:

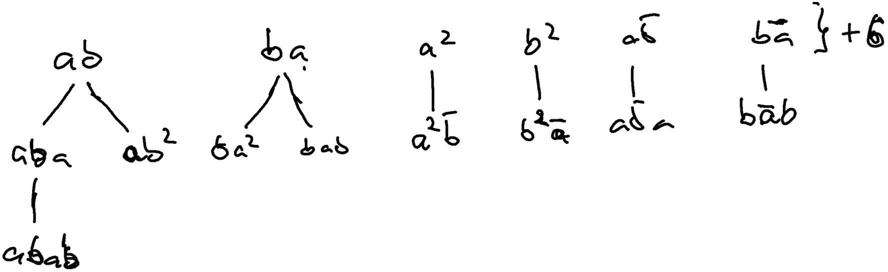
$$\begin{aligned} \alpha &= ababa + \sum S \\ \alpha^{-1} &= baba + \sum S \end{aligned}$$

where $S = \{ab, a^2, b^2, ba, a\overline{b}, b\overline{a}, \overline{a}b, \overline{a}^2, \overline{a}\overline{b}, \overline{b}a, \overline{b}\overline{a}, \overline{b}^2,$ (distance 2)
$aba, ab^2, a^2\overline{b}, b^2\overline{a}, ba^2, bab, a\overline{b}a, b\overline{a}b\}$.

- α and α^{-1} are rooted trees when adjoining $\{e, a, b, \overline{a}, \overline{b}\}$.
- S contains all the 12 elements of distance 2 from the root e.
- $\pi(S) = S$, and $\pi(\alpha) = \alpha^{-1}$ where $\pi \in \mathrm{Aut}(P)$ switches a and b.

The ball around e of radius 4 in Cayley graph of P has 41 elements. So there are 82 primary variables in the Boolean formula one has to satisfy.

Graphical representation of unit α with $\pi(\alpha) = \alpha^{-1}$



All nontrivial units supported on radius 4

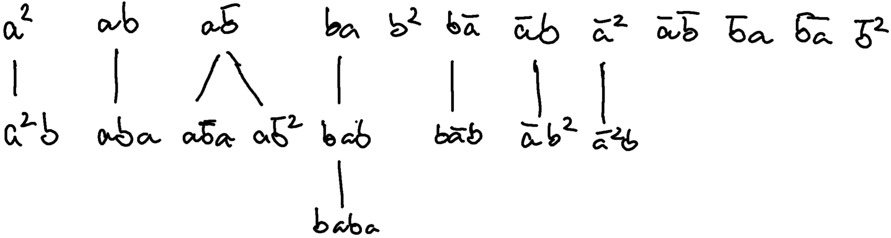
Recall again that we are working with the Hantzsche-Wendt group

$$P = \langle a, b \mid b^{-1}a^2b = a^{-2}, a^{-1}b^2a = b^{-2} \rangle,$$

and that $R = GF_2$.

- A SAT solver also verified that only the trivial units are supported on the ball of radius 3.
- Using the AllSAT solver of Guiseppe Spallitta (Trento), we found all the 18 nontrivial units supported on the ball around e of radius 4. (We don't distinguish α and α^{-1}.)
- All units and inverses have supports of size 21.
- Several of them satisfy $\alpha^{-1} = \pi(\alpha)$, where $\pi \in \mathrm{Aut}(P)$ switches a and b.

Another nontrivial unit in $GF_2[P]$ supported on radius 4



For the inverse, replace *baba* by *abab*.

II: The unique product property

Definition (Rudin and Schneider, 1964)

A torsion free group G has the unique product property (UPP) if for any nonempty finite sets $E, F \subseteq G$, some product r in the set EF is unique: $\exists r \; |\{\langle x, y\rangle \in E \times F \colon xy = r\}| = 1$.

Every one-sided orderable group has the UPP: e.g., $\mathbb{Z}, \mathrm{UT}_3(\mathbb{Z})$.

Proposition (Strownowski, 1980)

G has UPP $\Rightarrow$ G has TUP_R for each domain R.

- Proof: Suppose $\alpha, \beta \in R[G]$ satisfy $\alpha\beta = \beta\alpha = 1$, where α, β have supports A, B with $|A|, |B| \geq 2$.
- After translations, we can assume that $e \in A \cap B$.
- Let $E = B^{-1}A$ and $F = BA^{-1}$. Verify that there is NO unique product in EF.

UPP versus TUP

The studies of the two properties are closely connected.

- While UPP $\Rightarrow$ TUP_R, it is open whether the converse implication holds (even for a fixed domain R).

- Promislow (1988) showed that the same group
$$P = \langle a, b \mid b^{-1}a^2b = a^{-2}, a^{-1}b^2a = b^{-2} \rangle$$
fails the UPP via sets $E = F$ of 14 elements.

- As discussed, 32 years later, Gardam showed that UC fails over $GF_2[P]$, via support size 21.

The UPP expressed as a satisfiability problem

Given a finite set $S \subseteq G$, a satisfying truth assignment for the following formula yields sets $E, F \subseteq S$ failing the UPP.

Boolean variable a_s is true iff $s \in E$, and similarly for b_s and F.

$$\left(\bigvee_{s \in S} a_s\right) \wedge \left(\bigvee_{s \in S} b_s\right) \wedge \bigwedge_{u,v \in S} \left((a_u \wedge b_v) \to \bigvee_{\substack{u',v' \in S \\ u' \neq u \\ uv = u'v'}} (a_{u'} \wedge b_{v'})\right)$$

To rewrite this in conjunctive normal form (CNF), introduce auxiliary variables $c_{u,v}$ and impose the constraints that $c_{u,v} \leftrightarrow (a_u \wedge b_v)$.

We get the CNF formula

$$\bigwedge_{u,v \in S} \Big((\neg c_{u,v} \vee a_u) \wedge (\neg c_{u,v} \vee a_v) \wedge (\neg a_u \vee \neg a_v \vee c_{u,v})\Big)$$

$$\wedge \quad \Big(\bigvee_{s \in S} a_s\Big) \wedge \Big(\bigvee_{s \in S} b_s\Big) \wedge \bigwedge_{u,v \in S} \Big(\neg c_{u,v} \vee \bigvee_{\substack{u',v' \in S \\ u' \neq u \\ uv = u'v'}} c_{u',v'}\Big).$$

III. Fibonacci groups

The following is a source of examples for potential failure of UPP and TUP. For $2 \leq r < n$ let

$$F(r,n) = \langle\, x_1, \ldots, x_n \mid x_i x_{i+1} \cdots x_{i+r-1} = x_{i+r} \quad (1 \leq i \leq n)\,\rangle,$$

where subscripts are understood to be modulo n such that all x_j lie in $\{x_1, \ldots, x_n\}$. A nice fact is that

$$F(2,6) = \langle\, x_1, \ldots, x_6 \mid x_i x_{i+1} = x_{i+2} \quad (1 \leq i \leq 6)\,\rangle \cong P.$$

We are interested in $H_n := F(n-1, n)$ for even n because they are torsion-free and not left-orderable. Tietze transformations show

$$H_n = \langle x_1, \ldots, x_n \mid x_1^2 = \ldots = x_n^2 = w_n \rangle \quad \text{where} \quad w_n = x_1 \ldots x_n.$$

A new group failing the UPP

Using the SAT solver Kissat, we found that $F(3,4)$ fails the unique product property via the sets

$$E = \{1, x_1, x_4, x_1^{-1}, x_3^{-1}, x_1^2, x_1x_3, x_1x_2^{-1}, x_1x_3^{-1}, x_1x_4^{-1}, x_2x_1, x_2x_4^{-1}, x_3x_1^{-1}, x_4x_3, x_4x_1^{-1}, x_4x_2^{-1}, x_1^{-2}, x_1^{-1}x_2^{-1}, x_3^{-1}x_4^{-1}, x_1^3, x_1^2x_2, x_1x_3x_2^{-1}, x_1x_4x_1^{-1}, x_1x_4^{-1}x_2^{-1}, x_2x_1x_2, x_2x_4^{-1}x_2^{-1}, x_3x_1x_2^{-1}, x_3x_4^{-1}x_1^{-1}, x_3^{-1}x_4^{-1}x_2^{-1}\}$$

$$F = \{1, x_1, x_3, x_1^{-1}, x_3^{-1}, x_4^{-1}, x_1x_3^{-1}, x_1x_4^{-1}, x_2x_1, x_2x_4, x_2x_1^{-1}, x_2x_3^{-1}, x_3x_2^{-1}, x_3x_4^{-1}, x_4x_3, x_4x_2^{-1}, x_1^{-1}x_2^{-1}, x_2^{-1}x_3^{-1}, x_2^{-1}x_4^{-1}, x_1^2x_3, x_1^2x_4, x_2x_1x_2, x_2x_1x_3^{-1}, x_2x_4x_1^{-1}, x_2x_1^{-1}x_2^{-1}, x_2x_3^{-1}x_4^{-1}, x_2x_4^{-1}x_2^{-1}\}$$

Thus, each product in EF occurs at least twice.

Structure of $F(3,4)$

Using that $F(3,4) = \langle x_1, \ldots, x_4 \mid x_1^2 = \ldots = x_4^2 = x_1x_2x_3x_4 \rangle$, we obtained the power-conjugate presentation

$$F(3,4) = \mathrm{pc}\langle d, b, a, c \mid d^2 = c, \;\; b^d = b^{-1}, \;\; a^d = a^{-1}, \;\; a^b = ac^2 \rangle.$$

So $F(3,4)$ is

- an extension of index 4 of $U = \langle b, a, c^2 \rangle \cong \mathrm{UT}_3(\mathbb{Z})$;
- d acts by inverting a and b, and $\langle d^4 \rangle$ is the centre of U.

Since the structure of $F(3,4)$ is so easy, there is hope that it satisfies TUP_{GF_2}. This would show that $\mathrm{TUP}_{GF_2} \not\Rightarrow$ UPP .

Zero divisors conjecture (1)

- Kaplansky's 1959 zero divisors conjecture says that $R[G]$ has no zero divisors for any torsion free G and domain R.

- TUP_R implies that $R[G]$ has no zero divisors. That is, if there is a zero divisor, one can produce a nontrivial unit:

- First show that there is $\gamma \in R[G]$ such that $\gamma^2 = 0$.
  The unit is $\alpha = 1 + \gamma$ with inverse $\alpha^{-1} = 1 - \gamma$.

left orderable $\Rightarrow$ UPP $\Rightarrow$ TUP_R $\Rightarrow$ zero divisors_R

Zero divisors conjecture (2)

- The zero divisors conjecture holds for polycyclic G (Cliff, 1980)
- Using that the Heisenberg group is orderable, we have given a direct proof for $F(3,4)$.
- For $R = GF_2$, a unit that is an involution exists in $R[G]$ iff $R[G]$ has zero divisors.
- So our "almost involution" $\alpha \in R[P]^\times$ is as good as it gets.

We have shown that each $F(n-1,n)$ for even $n \geq 4$ has a solvable word problem. We plan to improve this to an efficient rewriting system.
Then we will try $F(5,6)$ as potential counterexamples to the zero divisors conjecture for $R = GF_2$.

Alan Reid pointed out that $F(2,n)$ is hyperbolic for $n \geq 8$, and so has the no zero divisors property.